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On a Porism Connected with the Theory of Maxwell's Equations and a Method of Obtaining the Lines of Electric Force Due to a Moving Point Charge

By H. Bateman.

§ 1. I have shown elsewhere* that when Maxwell's equations are written in Silberstein's concise form

$$\operatorname{rot} M = -\frac{i}{c}\frac{\partial M}{\partial t}, \qquad \operatorname{div} M = 0, \tag{1}$$

where M denotes the complex vector $H+iE$ and c is a constant, a solution may be obtained by writing for the three components of M

$$M_x = f(\alpha, \beta)\frac{\partial(\alpha,\beta)}{\partial(y,z)}, \quad M_y = f(\alpha,\beta)\frac{\partial(\alpha,\beta)}{\partial(z,x)}, \quad M_z = f(\alpha,\beta)\frac{\partial(\alpha,\beta)}{\partial(x,y)}, \tag{2}$$

where f is an arbitrary function of two quantities α, β which satisfy the partial differential equations †

$$\left.\begin{aligned} c\partial(\alpha, \beta | y, z) &= i\partial(\alpha, \beta | x, t), \\ c\partial(\alpha, \beta | z, x) &= i\partial(\alpha, \beta | y, t), \\ c\partial(\alpha, \beta | x, y) &= i\partial(\alpha, \beta | z, t). \end{aligned}\right\} \tag{3}$$

A general solution of these equations has already been obtained by taking x, y, α, β as new independent variables and transforming the equations. We shall now indicate a method by which the transformation may be performed very quickly.

It is easily seen that equations (3) can be written in the form

$$\left.\begin{aligned} c\partial(\alpha, \beta, x, t | y, z, x, t) &= i\partial(\alpha, \beta, y, z | x, t, y, z), \\ c\partial(\alpha, \beta, y, t | z, x, y, t) &= i\partial(\alpha, \beta, z, x | y, t, z, x), \\ c\partial(\alpha, \beta, z, t | x, y, z, t) &= i\partial(\alpha, \beta, x, y | z, t, x, y). \end{aligned}\right\} \tag{4}$$

Now multiply each Jacobian in equation (4) by $\partial(x, y, z, t | \alpha, \beta, x, y)$ and make use of the multiplication theorem for Jacobians. We then obtain a set of equations similar to (4) but with α, β, x, y as independent variables instead of x, y, z, t. The new equations reduce to the form

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* "The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equations," pp. 12, 122, 125, *Cambr. Univ. Press*; also *Bulletin of the American Mathematical Society*, Vol. XXI (1915), pp. 299–309. This paper will be cited later as "B."

† A different notation is now adopted for the Jacobians already used.

$$\frac{\partial t}{\partial y}=-\frac{i}{c}\frac{\partial z}{\partial x}, \qquad \frac{\partial t}{\partial x}=\frac{i}{c}\frac{\partial z}{\partial y}, \qquad \frac{\partial (z,\, t)}{\partial (x,\, y)}=\frac{i}{c}. \tag{5}$$

These equations may be solved as before, and the result is

$$z-ct=\phi+\theta(x+iy), \quad z+ct=\psi-\frac{1}{\theta}(x-iy), \tag{6}$$

where θ, ϕ, ψ are arbitrary functions of α and β.

Now let complex values be assigned to α and β and let x, y, z, t, be real. We proceed to find under what circumstances equations (6) can be satisfied. Write $\phi=\phi_1+i\phi_2$, $\psi=\psi_1+i\psi_2$, $\theta=\theta_1+i\theta_2$, where θ_1, θ_2, ϕ_1, ϕ_2, ψ_1, ψ_2 are all real; then, on equating the real and imaginary terms in equations (6), we find that

$$\left.\begin{aligned} z-ct&=\phi_1+\theta_1 x-\theta_2 y, & \phi_2+\theta_2 x+\theta_1 y&=0,\\ z+ct&=\psi_1-\frac{\theta_1 x-\theta_2 y}{\theta_1^2+\theta_2^2}, & \psi_2+\frac{\theta_2 x+\theta_1 y}{\theta_1^2+\theta_2^2}&=0. \end{aligned}\right\} \tag{7}$$

Hence, it appears that equations (6) can only be satisfied by real values of x, y, z, t, if the complex numbers α, β are chosen so that

$$\psi_2(\theta_1^2+\theta_2^2)=\phi_2. \tag{8}$$

When this condition is satisfied there are ∞^1 sets of values of x, y, z, t which satisfy equations (7). *Hence, the problem of finding real values of x, y, z, t corresponding to two complex numbers α, β is poristic.*

On solving equations (7) for x, y, z, we find that

$$\left.\begin{aligned} (1+v)z&=ct(1-v)+\phi_1+v\psi_1,\\ (1+v)y&=2\theta_2 ct+\theta_2(\phi_1-\psi_1)-\phi_2\theta_1(1+v^{-1}),\\ (1+v)x&=-2\theta_1 ct-\theta_1(\phi_1-\psi_1)-\phi_2\theta_2(1+v^{-1}), \end{aligned}\right\} \tag{9}$$

where $v=\theta_1^2+\theta_2^2$. *Hence, the different real values of x, y, z, t corresponding to a possible set of complex values of θ, ϕ, ψ are associated with a point which travels along a straight line with velocity c.*

We may regard the equations

$$X=\phi(\alpha,\, \beta), \quad Y=\psi(\alpha,\, \beta), \quad Z=\theta(\alpha,\, \beta) \tag{10}$$

as the equations of a surface S in the (X, Y, Z) space. Each point on this surface whose coordinates satisfy the condition (8) will correspond to a straight line which is described by a moving point with velocity c. Hence, with each surface S there is associated a complex of ∞^3 real straight lines described by points moving with velocity c. On account of the porism which has just been mentioned, one of the statements on p. 127 of my book is incorrect. It is easily seen that equations (280) on p. 124, viz.,

$$\left.\begin{aligned} (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2&=c^2(t-\tau)^2,\\ l(x-\xi)^2+m(y-\eta)^2+n(z-\zeta)^2&=c^2p(t-\tau),\\ l^2+m^2+n^2&=c^2p^2, \end{aligned}\right\} \tag{11}$$

can be thrown into the form (6). For, if we write $ict=s$, $ic\tau=\sigma$, and interpret x, y, z, s; ξ, η, ζ, σ as rectangular coordinates of points in a space of four dimensions, the first of equations (11) represents a hypersphere of zero radius, and the second, one of its tangent hyperplanes. The two equations consequently represent two planes, each of which passes through a generator of the sphere at infinity. Moreover, it is easy to see that the equations of such a plane can be thrown into the form (6). It is not correct, then, to say that if $\xi, \eta, \zeta, \tau, l, m, n, p$ are functions of two complex variables α, β, there are generally two real space-time points x, y, z, t corresponding to given values of α and β, for the special case to which I have referred on p. 127 is the only one which can arise.

The method which I have suggested for a specification of a real space-time point x, y, z, t by means of two complex numbers α, β thus has the disadvantage that there may be either no real point x, y, z, t or an infinite number of such points corresponding to a given pair of values of α and β.

§ 2. I have shown that a line of electric force in the electromagnetic field due to a moving point charge may be obtained by considering the positions at time t of a series of particles which are projected in certain directions from the different positions of the moving charge and travel along straight lines with the velocity of light. The direction cosines (l_0, m_0, n_0) of the direction of projection at time τ were found to satisfy three differential equations of type*

$$\mu\frac{dl_0}{d\tau}=\lambda\xi''+(\xi'-cl_0)(l_0\xi''+m_0\eta''+n_0\zeta''), \tag{12}$$

where $\mu=c^2-\xi'^2-\eta'^2-\zeta'^2$, $\lambda=c-l_0\xi'-m_0\eta'-n_0\zeta'$ and (ξ, η, ζ) denote the coordinates of the point charge at time τ. These equations are easily seen to be consistent with the relation $l_0^2+m_0^2+n_0^2=1$, for on multiplying the three equations by l_0, m_0, n_0, respectively, and adding, we find that $l_0\frac{dl_0}{d\tau}+m_0\frac{dm_0}{d\tau}+n_0\frac{dn_0}{d\tau}=0$.

Let us now write

$$ks=l_0+im_0, \quad k\sigma=l_0-im_0, \quad k=1+n_0. \tag{13}$$

Then it is easy to see that s satisfies the *Riccatian equation*

$$2\mu\frac{ds}{d\tau}=(\xi''+i\eta'')(c-\zeta')+\zeta''(\xi'+i\eta')-2s[c\zeta''+i(\xi'\eta''-\xi''\eta')]$$
$$+s^2[\zeta''(\xi'-i\eta')-(\zeta'+c)(\xi''-i\eta'')],$$

and that σ satisfies a similar equation in which $-i, \sigma$ are written in place of i and s respectively.

If these equations can be solved in any special case, l_0, m_0, n_0 can be expressed as functions of τ with the aid of equations (13), and then the coordinates of an arbitrary point on a line of electric force can be expressed as functions of the parameter τ.

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* B., p. 308. The primes are used to denote differentiation with regard to τ.